3

Quasiprobability distributions

3.1 Wigner representation

In classical optics the state of an electromagnetic oscillator is perfectly described by the statistics of the classical amplitude α. The amplitude may be completely fixed (then the field is coherent), or α may fluctuate (then the field is partially coherent or incoherent). In classical optics as well as in classical mechanics, we can characterize the statistics of the complex amplitude α or, equivalently, the statistics of the components position q and momentum p introducing a phase-space distribution $W(q, p)$. (As explained in Section 2.1, the real and the imaginary part of the complex amplitude α can be regarded as the position and the momentum of the electromagnetic oscillator.) The distribution $W(q, p)$ quantifies the probability of finding a particular pair of q and p values in their simultaneous measurement. Knowing the phase-space probability distribution, all statistical quantities of the electromagnetic oscillator can be predicted by calculation. In this sense the phase-space distribution describes the state in classical physics. All this is much more subtle in quantum mechanics. First of all, Heisenberg's uncertainty principle prevents us from observing position and momentum simultaneously *and* precisely. So it seems there is no point in thinking about quantum phase space. But wait! In quantum mechanics we cannot directly observe quantum states either. Nevertheless, we are legitimately entitled to use the concept of states as if they were existing entities (whatever they are). We use their properties to predict the statistics of observations. Why not use a quantum phase-space distribution $W(q, p)$ solely to calculate observable quantities in a classicallike fashion? Clearly, the concept of quantum phase space must contain a certain flaw. The probability distribution $W(q, p)$ could become negative, for instance, or ill-behaved. Also, the classicallike the fashion of making statistical predictions may seem to be classicallike at the first glance but not at the second. For these very reasons we should call $W(q, p)$ a *quasiprobability distribution*. Furthermore, there are certainly infinitely many

ways of making up quasiprobability distributions (simply because there is no way of defining them properly). Which one shall we choose? Is there a royal road to quantum phase space?

3.1.1 *Wigner's formula*

Bertrand and Bertrand [29] had the brilliant idea of defining the quasiprobability distribution $W(q, p)$ by postulating its properties. Just one postulate turns out to be sufficient. Let us assume that the distribution $W(q, p)$ *behaves* like a joint probability distribution for q and p without ever mentioning any simultaneous observation of position and momentum. What can we say about classical probability distributions? The *marginal distributions* or, in other words, the reduced distributions $\int_{-\infty}^{+\infty} W(q, p)\, dp$ or $\int_{-\infty}^{+\infty} W(q, p)\, dq$ must yield the position or the momentum distribution, respectively. Additionally, if we perform a phase shift θ all complex amplitudes are shifted in phase, meaning that the components q and p rotate in the two-dimensional phase space (q, p). A classical probability distribution for position and momentum values would rotate accordingly. In view of this fact we postulate that the position probability distribution $\mathrm{pr}(q, \theta)$ after an arbitrary phase shift θ should equal

$$\begin{aligned}\mathrm{pr}(q, \theta) &\equiv \langle q|\hat{U}(\theta)\hat{\rho}\hat{U}^\dagger(\theta)|q\rangle \\ &= \int_{-\infty}^{+\infty} W(q\cos\theta - p\sin\theta, q\sin\theta + p\cos\theta)\, dp. \qquad (3.1)\end{aligned}$$

This single formula marries the quasiprobability distribution $W(q, p)$ with quantum mechanics. The same formula ties $W(q, p)$ to observable quantities. And, even more remarkably, the formula links quantum states to observations. Considering special cases of formula (3.1) we see that the marginal distributions of $W(q, p)$ produce the correct position and momentum probabilities, respectively. For $\theta = 0$ we obtain

$$\int_{-\infty}^{+\infty} W(q, p)\, dp = \langle q|\hat{\rho}|q\rangle \qquad (3.2)$$

and for $\theta = \pi/2$

$$\langle p|\hat{\rho}|p\rangle = \int_{-\infty}^{+\infty} W(-q, p)\, dq = \int_{-\infty}^{+\infty} W(q, p)\, dq. \qquad (3.3)$$

[To justify Eq. (3.3) we note that $\hat{U}^\dagger(\pi/2)|q\rangle$ is a momentum eigenstate $|p = q\rangle$ with eigenvalue q according to Eq. (2.15).] Integrals such as (3.1) are called *Radon transformations* [226], and they are thoroughly studied in the mathematical theory of tomographic imaging [113], [194]. The inversion of the Radon transformation [226] plays a distinguished role in tomography. In quantum-state tomography the *inverse Radon transformation* turns out to be the mathematical key for quantum-state reconstruction. See Section 5.1.1.

Why is postulate (3.1) sufficient? To understand the reason we introduce the Fourier-transformed distribution $\tilde{W}(u, v)$ called the *characteristic function*

$$\tilde{W}(u, v) \equiv \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} W(q, p)\exp(-iuq - ivp)\,dq\,dp \tag{3.4}$$

and the Fourier-transformed position probability distribution $\widetilde{\mathrm{pr}}(\xi, \theta)$

$$\widetilde{\mathrm{pr}}(\xi, \theta) \equiv \int_{-\infty}^{+\infty} \mathrm{pr}(q, \theta)\exp(-i\xi q)\,dq. \tag{3.5}$$

On the other hand, the basic postulate (3.1) for $W(q, p)$ requires that

$$\widetilde{\mathrm{pr}}(\xi, \theta) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} W(q', p')\exp(-i\xi q)\,dq\,dp, \tag{3.6}$$

with the abbreviations

$$q' = q\cos\theta - p\sin\theta \quad \text{and} \quad p' = q\sin\theta + p\cos\theta. \tag{3.7}$$

Consequently, q is given by

$$q = q'\cos\theta + p'\sin\theta. \tag{3.8}$$

We change the integration variables from (q', p') to (q, p) and obtain, according to the very definition of the characteristic function (3.4),

$$\widetilde{\mathrm{pr}}(\xi, \theta) = \tilde{W}(\xi\cos\theta, \xi\sin\theta). \tag{3.9}$$

The Fourier-transformed position probability distribution is the characteristic function in polar coordinates. In this way the two functions are intimately related. So far we have used only the second line of the fundamental postulate (3.1) or, so to say, the classical nature of the quasiprobability distribution $W(q, p)$. The quantum features come into play when we substitute the first line, that is, the definition of $\mathrm{pr}(q, \theta)$ in the Fourier transformation (3.5). We obtain explicitly

$$\begin{aligned}
\widetilde{\mathrm{pr}}(\xi, \theta) &= \int_{-\infty}^{+\infty} \langle q|\hat{U}(\theta)\hat{\rho}\hat{U}^\dagger(\theta)|q\rangle \exp(-i\xi q)\,dq \\
&= \int_{-\infty}^{+\infty} \langle q|\hat{U}(\theta)\hat{\rho}\hat{U}^\dagger(\theta)\exp(-i\xi\hat{q})|q\rangle\,dq \\
&= \mathrm{tr}\{\hat{U}(\theta)\hat{\rho}\hat{U}^\dagger(\theta)\exp(-i\xi\hat{q})\} \\
&= \mathrm{tr}\{\hat{\rho}\hat{U}^\dagger(\theta)\exp(-i\xi\hat{q})\hat{U}(\theta)\}.
\end{aligned} \tag{3.10}$$

We use the rotation formula (2.15) for the quadrature operators to get

$$\hat{U}^\dagger(\theta)\exp(-i\xi\hat{q})\hat{U}(\theta) = \exp(-i\hat{q}\xi\cos\theta - i\hat{p}\xi\sin\theta). \tag{3.11}$$

This is the *Weyl operator* [289] $\exp(-iu\hat{q} - iv\hat{p})$ in polar coordinates. The Fourier transform $\widetilde{\mathrm{pr}}(\xi, \theta)$ gives the characteristic function in polar coordinates, as we have learned from Eq. (3.9). Consequently,

$$\tilde{W}(u, v) = \mathrm{tr}\{\hat{\rho}\exp(-iu\hat{q} - iv\hat{p})\}. \tag{3.12}$$

The characteristic function is the "quantum Fourier transform" of the density operator. Because the characteristic function $\tilde{W}(u, v)$ is the Fourier transform of $W(q, p)$ by definition, the quasiprobability distribution $W(q, p)$ should be very closely related to the density operator. Indeed, both are one-to-one representations of the quantum state, as we will show in the next subsection, 3.1.2. Let us calculate the trace in Eq. (3.12) in the position representation. We use the Baker–Hausdorff formula (2.55) to reexpress the Weyl operator

$$\exp(-iu\hat{q} - iv\hat{p}) = \exp\left(-i\frac{uv}{2}\right)\exp(-iu\hat{q})\exp(-iv\hat{p}). \tag{3.13}$$

The operator $\exp(-iv\hat{p})$ shifts the position eigenstates $|q\rangle$ by v to produce $|q + v\rangle$ because of the relation (2.21) between the position and momentum eigenstates. Consequently,

$$\begin{aligned}
\tilde{W}(u, v) &= \int_{-\infty}^{+\infty} \langle q|\hat{\rho}\exp(-iu\hat{q} - iv\hat{p})|q\rangle\,dq \\
&= \exp\left(-i\frac{uv}{2}\right)\int_{-\infty}^{+\infty} \langle q|\hat{\rho}\exp(-iuq)|q + v\rangle\,dq.
\end{aligned} \tag{3.14}$$

We replace q by $x - v/2$ and obtain the compact formula

$$\tilde{W}(u, v) = \int_{-\infty}^{+\infty} \exp(-iux)\left\langle x - \frac{v}{2}\middle|\hat{\rho}\middle|x + \frac{v}{2}\right\rangle dx. \tag{3.15}$$

To derive an explicit expression for the quasiprobability distribution $W(q, p)$, we simply invert the Fourier transformation in definition (3.4) and get by virtue of formula (3.15) for the characteristic function

$$\begin{aligned}
W(q, p) &= \frac{1}{(2\pi)^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \tilde{W}(u, v)\exp(iuq + ivp)\,du\,dv \\
&= \frac{1}{(2\pi)^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \left\langle q' - \frac{v}{2}\middle|\hat{\rho}\middle|q' + \frac{v}{2}\right\rangle \\
&\quad \times \exp(-iuq' + iuq + ivp)\,dq'\,du\,dv \\
&= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \left\langle q' - \frac{v}{2}\middle|\hat{\rho}\middle|q' + \frac{v}{2}\right\rangle \exp(ivp) \\
&\quad \times \delta(q' - q)\,dv\,dq'.
\end{aligned} \tag{3.16}$$

Setting $x = v$ we obtain, finally,

$$W(q,p) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \exp(ipx)\left\langle q - \frac{x}{2}\right|\hat{\rho}\left|q + \frac{x}{2}\right\rangle dx. \tag{3.17}$$

This is Wigner's legendary formula [290] for a classicallike phase-space distribution in quantum mechanics called the *Wigner function*. It appeared for the first time in his 1932 paper [290] "On the Quantum Correction for Thermodynamic Equilibrium." It was "chosen from all possible expressions, because it seems to be the simplest."*

3.1.2 Basic properties

Wigner's representation of quantum mechanics has found many applications in broad areas of quantum physics. It was used whenever quantum corrections to classical laws were of interest, as in Wigner's 1932 paper [290]. Countless articles have been written on the Wigner function itself. Because there is room neither for mentioning all of them here nor for deriving all known properties of the Wigner function, the reader is referred to Moyal's early yet excellent text [191] and to the more recent review articles by Tatarskii [263]; Balazs and Jennings [10]; Hillery, O'Connell, Scully, and Wigner [116]; and Lee [149] and to the book [135] by Kim and Noz. Here we will focus on only the basic properties of the Wigner function. First, we note that the Wigner function is real

$$W^*(q,p) = W(q,p) \tag{3.18}$$

for Hermitian operators $\hat{\rho}$. This property is verified by considering the complex conjugate of Wigner's formula (3.17) and replacing x by $-x$. The Wigner function is normalized

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} W(q,p)\,dq\,dp = 1, \tag{3.19}$$

as is easily seen from Wigner's formula (3.17), because the density operator $\hat{\rho}$ is normalized so that $\operatorname{tr}\{\hat{\rho}\} = 1$. So far the Wigner function shows features of a proper probability distribution.

A remarkable property of the Wigner representation is the *overlap formula*

$$\operatorname{tr}\{\hat{F}_1\hat{F}_2\} = 2\pi\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} W_1(q,p)W_2(q,p)\,dq\,dp \tag{3.20}$$

for the Wigner functions W_1 and W_2 of two arbitrary operators $\hat{F}_1$ and $\hat{F}_2$. Both operators are not even required to be Hermitian, and we have used Wigner's

*A footnote, however, says that "this expression was found by L. Szilard and [E.P. Wigner] some years ago for another purpose . . . ."

formula (3.17) with $\hat{F}_k$ instead of $\hat{\rho}$ to define a Wigner function for the $\hat{F}_k$ operators. The proof of the overlap formula (3.20) is a straightforward calculation using Wigner's expression (3.17) for the right-hand side

$$\begin{aligned}
&\frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \exp[ip(x_1+x_2)]\left\langle q - \frac{x_1}{2}\right|\hat{F}_1\left|q + \frac{x_1}{2}\right\rangle \\
&\quad\times\left\langle q - \frac{x_2}{2}\right|\hat{F}_2\left|q + \frac{x_2}{2}\right\rangle dx_1\,dx_2\,dq\,dp \\
&= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left\langle q - \frac{x}{2}\right|\hat{F}_1\left|q + \frac{x}{2}\right\rangle\left\langle q + \frac{x}{2}\right|\hat{F}_2\left|q - \frac{x}{2}\right\rangle dq\,dx \\
&= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\langle q'|\hat{F}_1|q''\rangle\langle q''|\hat{F}_2|q'\rangle\,dq'\,dq'' \\
&= \int_{-\infty}^{+\infty}\langle q'|\hat{F}_1\hat{F}_2|q'\rangle\,dq' \\
&= \operatorname{tr}\{\hat{F}_1\hat{F}_2\}.
\end{aligned} \tag{3.21}$$

Why is the overlap formula remarkable? We can use it for calculating expectation values

$$\operatorname{tr}\{\hat{\rho}\hat{F}\} = 2\pi\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} W(q,p)W_F(q,p)\,dq\,dp. \tag{3.22}$$

(We have simply replaced $\hat{F}_1$ by $\hat{\rho}$ and $\hat{F}_2$ by $\hat{F}$.) This equation would be the rule for predicting expectations in classical statistical physics, too. The Wigner function $W(q,p)$ plays the role of a classical phase-space density, whereas $W_F(q,p)$ appears as the physical quantity that is averaged with respect to $W(q,p)$. This is exactly the classicallike fashion of calculating quantum-mechanical expectation values we were seeking. We can understand formula (3.22) another way by seeing $W_F(q,p)$ as a filter function. Consequently, all that quantum mechanics allows us to predict are filtered projections of the Wigner function. All that we can see are shadows of the states, very much in the sense of Plato's famous parable [219] mentioned in the introduction, yet formulated more precisely, that is, more quantitatively.

Another simple consequence of the overlap formula (3.20) is the expression

$$|\langle\psi_1\,|\,\psi_2\rangle|^2 = 2\pi\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} W_1(q,p)W_2(q,p)\,dq\,dp \tag{3.23}$$

for the transition probability between the pure states $|\psi_1\rangle$ and $|\psi_2\rangle$. However, this quantity vanishes if the states $|\psi_1\rangle$ and $|\psi_2\rangle$ are orthogonal

$$\langle\psi_1\,|\,\psi_2\rangle = 0. \tag{3.24}$$

The overlap (3.23) of two positive functions W_1 and W_2 cannot be zero. Consequently, Wigner functions cannot be positive in general. (In fact, states having Gaussian wave functions are the only pure states with nonnegative Wigner functions. See [119], [185], and [263].) In this way the overlap formula (3.20) reveals strikingly both the similarities and the differences between a classical probability distribution and the Wigner function. Quantum interference implies that the classicallike Wigner function cannot be regarded as a probability distribution but as a quasiprobability distribution only. This property is one way in which the unavoidable flaw in the concept of quantum phase space may appear. Negative regions in the Wigner function of a given state can be seen as signatures of nonclassical behavior [185].

Using the overlap formula (3.20) we are also able to quantify the purity of a quantum state. In fact, identifying both $\hat{F}_1$ and $\hat{F}_2$ with $\hat{\rho}$ we obtain

$$\operatorname{tr}\{\hat{\rho}^2\} = 2\pi \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} W(q,p)^2\, dq\, dp. \tag{3.25}$$

The *purity* $\operatorname{tr}\{\hat{\rho}^2\}$ ranges between zero and unity and equals exactly unity if and only if the state is pure ($\hat{\rho} = |\psi\rangle\langle\psi|$). According to relation (1.21) the von-Neumann entropy S is bounded by

$$S \equiv -\operatorname{tr}\{\hat{\rho}\ln\hat{\rho}\} \geq 1 - 2\pi \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} W(q,p)^2\, dq\, dp. \tag{3.26}$$

We see that the overlap of the Wigner function with itself provides a convenient way of expressing statistical purity in quantum mechanics.

Finally, we can use the overlap formula (3.20) to represent the density-matrix elements in a given basis in terms of the Wigner function

$$\langle a'|\hat{\rho}|a\rangle = \operatorname{tr}\{\hat{\rho}|a\rangle\langle a'|\} = 2\pi \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} W(q,p) W_{a'a}(q,p)\, dq\, dp. \tag{3.27}$$

Here $W_{a'a}(q,p)$ denotes the Wigner representation of the projector $|a\rangle\langle a'|$ obtained by replacing $\hat{\rho}$ by $|a\rangle\langle a'|$ in Wigner's formula (3.17). This property shows that the Wigner function is indeed a one-to-one representation of the quantum state.

We may turn the tables and ask, is any normalized real function $W(q,p)$ always a Wigner function, that is, does it correspond to a state? Obviously it does not, because the integral of the squared function must be less than or equal to $(2\pi)^{-1}$ according to the purity relation (3.25). Another quantum constraint is imposed on any realistic Wigner function. The values of it may range between only $\pm\pi^{-1}$, that is

$$|W(q,p)| \leq \frac{1}{\pi}. \tag{3.28}$$

To prove this inequality we consider a pure state $\hat{\rho} = |\psi\rangle\langle\psi|$ first. We use the Schwarz inequality to estimate the Wigner function given in terms of Wigner's formula (3.17) and obtain

$$|W(q,p)|^2 \leq \frac{1}{(2\pi)^2}\int_{-\infty}^{+\infty}\left|\left\langle q-\frac{x}{2}\middle|\psi\right\rangle\right|^2 dx \int_{-\infty}^{+\infty}\left|\left\langle q+\frac{x}{2}\middle|\psi\right\rangle\right|^2 dx = \frac{1}{\pi^2} \tag{3.29}$$

because the state vector $|\psi\rangle$ is normalized. In case of a statistical mixture the density matrix can be represented as a sum of pure states $|\psi_a\rangle\langle\psi_a|$ weighted by their probabilities ρ_a according to the very definition (1.12) of the density operator. Consequently, the Wigner function for a mixed state is a weighted sum of pure Wigner functions as well. By estimating the individual pure Wigner functions and summing with respect to the normalized probabilities p_a, we see easily that the bound (3.28) is valid for mixed states, too. [Note that the Wigner function $W_n(q,p)$ for Fock states $|n\rangle$ actually equals $(-1)^n/\pi$ at the origin $q = p = 0$; see Eq. (3.83).] The constraint (3.28) shows that Wigner functions cannot be highly peaked, meaning that the quantum "phase-space density" cannot be arbitrarily high and Wigner functions cannot approach delta functions $\delta(q-q_0)\delta(p-p_0)$, for instance. Of course, according to Heisenberg's uncertainty principle, position and momentum must fluctuate statistically, and this intrinsic uncertainty is mirrored in the uniform bound (3.28). Note that other constraints on Wigner functions were given by Tatarskii [263] and Lieb [175]. However, no golden rule decides whether a given function is a Wigner function, apart from the Solomonic statement that any density matrix derived from a proper Wigner function should be a density matrix, or have nonnegative eigenvalues. Equivalently, all main-diagonal elements $\langle a|\hat{\rho}|a\rangle$ derived according to formula (3.27) must be nonnegative. Deviations from this law indicate imperfections in experimentally reconstructed Wigner functions, for instance.

Apart from formula (3.22) another equivalent way exists of making quantum-mechanical predictions, that is, of calculating expectation values via the Wigner function. We consider

$$\begin{aligned}\operatorname{tr}\{\hat{\rho}(\lambda\hat{q}+\mu\hat{p})^k\} &= \mathrm{i}^k \frac{\partial^k}{\partial\xi^k}\tilde{W}(\xi\lambda,\xi\mu)\bigg|_{\xi=0} \\ &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} W(q,p)(\lambda q+\mu p)^k\, dq\, dp. \end{aligned}\tag{3.30}$$

In the first line we have used key formula (3.12) for the characteristic function $\tilde{W}(u,v)$, whereas in the second line we have used the Fourier relationship (3.4) between $\tilde{W}(u, v)$ and the Wigner function. Comparing the powers of λ and μ we see that

$$\mathrm{tr}\{\hat{\rho}\mathcal{S}\hat{q}^m\hat{p}^n\} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} W(q, p) q^m p^n \, dq \, dp. \tag{3.31}$$

The symbol $\mathcal{S}$ means that we should symmetrize all possible products of the m $\hat{q}$-operators and the n $\hat{p}$-operators, that is, we should take the average over all products with the right amount of $\hat{q}$'s and $\hat{p}$'s. So, for example, $1/3$ $(\hat{q}^2\hat{p} + \hat{q}\hat{p}\hat{q} + \hat{p}\hat{q}^2)$ corresponds to $q^2 p$. This *Weyl correspondence* [289] is also a convenient way of making quantum-mechanical predictions in a classicallike fashion. [Note that the Weyl correspondence is completely equivalent to formula (3.22).] Given a symmetrized operator $\hat{F}$, we can calculate quantum-mechanical averages as if $\hat{F}$ were a classical quantity. However, this pleasing property is Janus-faced. The square of $\hat{F}$, which describes the fluctuations of $\hat{F}$, is not necessarily symmetrized [and the Wigner function of $\hat{F}^2$ is not always $W_F(q, p)^2$]. We should express $\hat{F}^2$ in terms of symmetrized operators to get meaningful results. This route is another way in which the mutual exclusion of certain observables sneaks in via the commutator relations of position and momentum. So we must not forget that the algebraic structures of quantum mechanics and classical physics are different, despite many similarities. This difference causes a problem in the very concept of a quantum phase space even more serious than negative "probabilities."

3.1.3 Examples

How do typical Wigner functions look? Are they similar to classical phase-space densities? Probably the simplest example is the Wigner function for the vacuum state. We insert the quadrature wave function (2.33) in Wigner's formula (3.17) and see that the Wigner function for a vacuum is Gaussian

$$W_0(q, p) = \frac{1}{\pi}\exp(-q^2 - p^2). \tag{3.32}$$

Classically, this function would correspond to the phase-space density of an ensemble of electromagnetic oscillators fluctuating statistically around the origin in phase space with isotropic variances of $1/2$ in our units. Quantum-mechanically, these statistical fluctuations occur even if the spatial–temporal mode is in a pure vacuum. Figure 3.1 shows the experimentally reconstructed Wigner function for a vacuum, illustrating beautifully the isotropic character of the vacuum fluctuations (except for small experimental errors). How does a

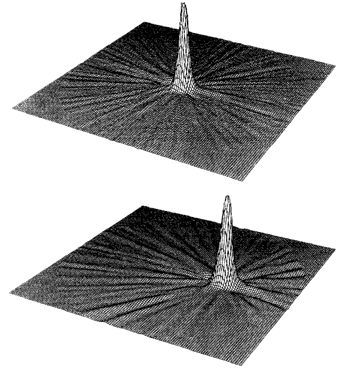

Fig. 3.1. Wigner function for a vacuum (top) and for a coherent state (bottom). We see clearly that coherent states are just "displaced vacuums." Optical homodyne tomography (see Chapter 5) was used to reconstruct the Wigner functions from experimental data. [Courtesy of G. Breitenbach, University of Constance.]

squeezed vacuum look? Let us study the general effect of squeezing in phase space first. We obtain from Wigner's formula (3.17)

$$\begin{aligned} W_s(q, p) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \exp(ipx)\left\langle q - \frac{x}{2}\right|\hat{S}\hat{\rho}\hat{S}^\dagger\left|q + \frac{x}{2}\right\rangle dx \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \exp(ipx)e^\zeta\left\langle e^\zeta\left(q - \frac{x}{2}\right)\right|\hat{\rho}\left|e^\zeta\left(q + \frac{x}{2}\right)\right\rangle dx \end{aligned} \tag{3.33}$$

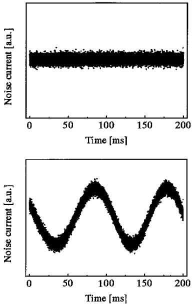

Fig. 3.2. Quadrature fluctuations of a vacuum (top) and a coherent state (bottom). Homodyne detection (see Section 4.2) was employed for performing the quadrature measurements on stationary fields. Each point represents a single measurement result. The reference phase (the phase of the local oscillator) was gradually shifted during the observation time. We see clearly the phase independence of the vacuum fluctuations and the oscillating effect, Eq. (2.47), of phase shifts on the quadratures of a coherent state. The plot shows a part of the experimental data used to reconstruct the Wigner functions of the previous figure via optical homodyne tomography (see Chapter 5). [Courtesy of G. Breitenbach, University of Constance.]

because the squeezing operator $\hat{S}$ defined in Eq. (2.82) rescales the position wave function according to Eq. (2.79). We substitute $e^{\zeta}x$ with x' and get the result

$$W_s(q, p) = W(e^{\zeta}q, e^{-\zeta}p). \tag{3.34}$$

The Wigner function for a squeezed state is squeezed in one quadrature direction and stretched accordingly in the orthogonal line in order to preserve the area in phase space. In this way the quadrature fluctuations displayed in the Wigner function are redistributed from one quadrature to the canonically conjugate quantity. This redistribution is exactly what we would expect from squeezing in

phase space. Using the general result (3.34), we obtain directly from Eq. (3.32) the Wigner function of a squeezed vacuum

$$W_s(q, p) = \frac{1}{\pi}\exp(-e^{2\zeta}q^2 - e^{-2\zeta}p^2). \tag{3.35}$$

As for a vacuum, the Wigner function is a Gaussian distribution with, however, unbalanced variances (2.78) indicating the effect of quadrature squeezing. Figure 3.3 shows the experimentally reconstructed Wigner function of a significantly squeezed vacuum generated by parametric amplification. It is an easy exercise to calculate the distribution $\mathrm{pr}(q, \theta)$ for phase-shifted quadratures from the Wigner function (3.35) via the Radon transformation (3.1). We

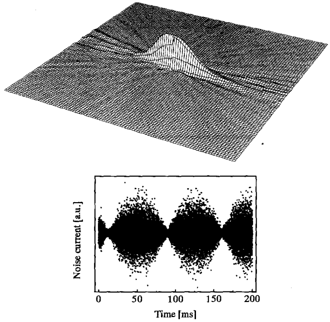

Fig. 3.3. Squeezed vacuum. Wigner function (top) and quadrature fluctuations (bottom). We see clearly a remarkable squeezing in phase space and the corresponding breathing of the quadrature noise, Eq. (3.37). The noise trace shows a part of the experimental data used to reconstruct the depicted Wigner function via optical homodyne tomography (see Chapter 5). [Courtesy of G. Breitenbach, University of Constance.] See also Ref. [41].

find the result

$$\mathrm{pr}(q,\theta) = \left(2\pi\Delta_\theta^2 q\right)^{-1/2} \exp\left(-\frac{q^2}{2\Delta_\theta^2 q}\right) \tag{3.36}$$

with the phase-dependent variance

$$\Delta_\theta^2 q = \frac{1}{2}(\mathrm{e}^{2\zeta}\sin^2\theta + \mathrm{e}^{-2\zeta}\cos^2\theta). \tag{3.37}$$

The quadrature fluctuations of a squeezed vacuum are Gaussian and, of course, phase dependent. Their variances $\Delta_\theta^2 q$ vary from $\frac{1}{2}\mathrm{e}^{-2\zeta}$ to $\frac{1}{2}\mathrm{e}^{+2\zeta}$ with a period of π.

What is the Wigner function of a coherent state? Coherent states are displaced vacuums, so we would expect that their Wigner functions are displaced vacuum Wigner functions, too, with a displacement given by the complex coherent amplitude $2^{1/2}\alpha = q_0 + \mathrm{i}p_0$. That this expectation is correct is easily seen, considering the general effect of the displacement operator $\hat{D}$ in phase space

$$\begin{aligned} W_D(q,p) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \exp(\mathrm{i}px)\left\langle q-\frac{x}{2}\right|\hat{D}\hat{\rho}\hat{D}^\dagger\left|q+\frac{x}{2}\right\rangle dx \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \exp[\mathrm{i}(p-p_0)x] \\ &\quad\times\left\langle q-\frac{x}{2}-q_0\right|\hat{\rho}\left|q+\frac{x}{2}-q_0\right\rangle dx \end{aligned} \tag{3.38}$$

according to the general rule (2.56) and (2.59) for displacing position wave functions. We find that Wigner functions of displaced states are indeed just displaced Wigner functions

$$W_D(q,p) = W(q-q_0, p-p_0) \tag{3.39}$$

and, consequently, the Wigner function of a coherent state is given by the displaced Gaussian distribution

$$W(q,p) = \frac{1}{\pi}\exp[-(q-q_0)^2 - (p-p_0)^2]. \tag{3.40}$$

Again, the Wigner function displays the typical features of the considered quantum state: A coherent state as produced by a high-quality laser has a stable coherent amplitude $q_0 + \mathrm{i}p_0$ that is contaminated by the unavoidable vacuum fluctuations only.

According to the fundamental superposition principle of quantum mechanics, we are entitled to think of quantum superpositions of coherent states. These are states that contain simultaneously two coherent components, one pointing in one direction in phase space and the other pointing in another. The position wave function ψ of such a state would be the superposition of two coherent-state

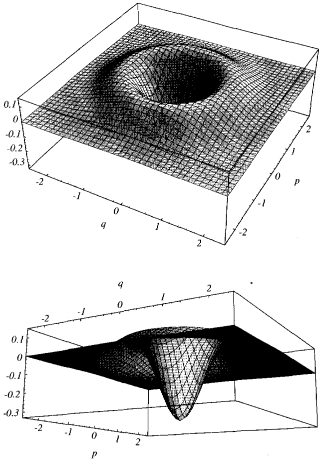

Fig. 3.4. Wigner function of a single photon (a one-photon Fock state) seen from above and from below. According to Eq. (3.83), the Wigner function is given by the expression $W(q,p) = \exp(-q^2-p^2)(2q^2+2p^2-1)/\pi$. Negative "probabilities" are clearly visible near the origin of the phase space.

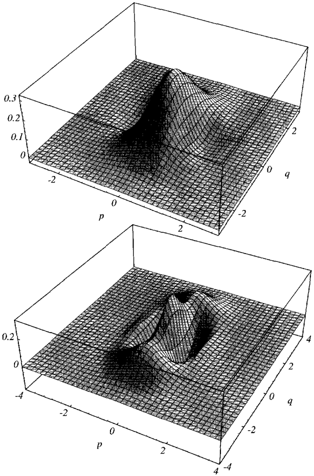

Fig. 3.5. Wigner function of Schrödinger-cat states defined in Eq. (3.41). Top: $q_0 = 1$. The "cat" shows only a mere squeezing instead of a clear separation of two coherent amplitudes. Bottom: $q_0 = 2$. First indications of two distinct humps are visible in the quantum-interference structure.

wave functions; for instance,

$$\psi(q) \propto \exp\left[-\frac{1}{2}(q - q_0)^2\right] + \exp\left[-\frac{1}{2}(q + q_0)^2\right]. \qquad (3.41)$$

(The normalization factor is not important here and has been omitted.) The wave function shows two peaks, one at q_0 and the other at $-q_0$ according to the superimposed coherent amplitudes. Note that this quantum superposition (3.41) has nothing to do with optical interference. When two fields interfere, their amplitude may be enhanced or canceled, producing, for example, coherent states of enhanced or zero amplitude (vacuum). The quantum superposition (3.41) still contains both coherent amplitudes $\pm q_0$. It is also much different from an incoherent superposition of $\pm q_0$, where the field has either the amplitude $+q_0$ or the amplitude $-q_0$ with certain probabilities. The quadrature amplitude of the superposition state (3.41) is $+q_0$ *as well as* $-q_0$, with a resolution given by the vacuum fluctuations. This strange behavior of being at $+q_0$ as well as at $-q_0$ resembles Schrödinger's famous Gedanken experiment about a quantum cat being simultaneously alive and dead [246]. Therefore, states such as (3.41) are named *Schrödinger-cat states*. They have not yet been observed in the optical domain, because they are extremely vulnerable to losses. Quantum decoherence [308] caused by linear losses is the main reason that extremely strange quantum phenomena allowed in theory are very difficult to observe in practice. Which observable phenomena of the Schrödinger-cat state (3.41) would we expect? We calculate the Wigner function using Wigner's formula (3.17) and obtain

$$\begin{aligned} W(q, p) \propto\ & \exp[-(q - q_0)^2 - p^2] + \exp[-(q + q_0)^2 - p^2] \\ & + 2\exp(-q^2 - p^2)\cos(2pq_0). \end{aligned} \qquad (3.42)$$

Like the wave function, the Wigner function exhibits two peaks at the coherent amplitudes $\pm q_0$. However, the interference structure halfway between the peaks displays the quantum superposition of both amplitudes, showing rapid oscillations with a frequency given by the distance $2q_0$ of the superimposed amplitudes. See Fig. 3.6. The Wigner function becomes negative, indicating the nonclassical behavior of the Schrödinger-cat state [47], [243]. To predict observable effects of the quantum-superposition state (3.41) we calculate the quadrature distributions $\mathrm{pr}(q, \theta)$ via Radon transformation (3.1) of the Wigner function (3.42) and get

$$\begin{aligned} \mathrm{pr}(q, \theta) \propto\ & \exp[-(q - q_0\cos\theta)^2] + \exp[-(q + q_0\cos\theta)^2] \\ & + 2\exp\left(-q^2 - q_0^2\cos^2\theta\right)\cos(2qq_0\sin\theta). \end{aligned} \qquad (3.43)$$

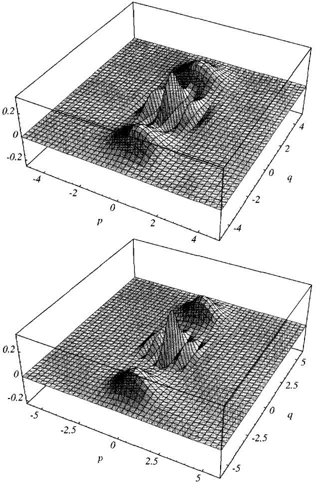

Fig. 3.6. Wigner function of Schrödinger-cat states defined in Eq. (3.41). Top: $q_0 = 3$. Two separated coherent amplitudes are clearly visible. Bottom: $q_0 = 4$. The larger the separation of the amplitudes, the more rapid is the oscillation in the quantum-interference structure.

Shifting the phase θ turns the position quadrature distribution

$$\mathrm{pr}(q) \approx \frac{1}{2}\exp[-(q-q_0)^2] + \frac{1}{2}\exp[-(q+q_0)^2] \tag{3.44}$$

(showing peaks at $\pm q_0$) into the momentum distribution

$$\mathrm{pr}(p) \propto \exp(-p^2)\cos^2(pq_0) \tag{3.45}$$

at $\theta = \pi/2$ and $q = p$, displaying typical interference fringes. The interference pattern is mirrored in the highly oscillating Wigner function of a Schrödinger-cat state.

We have seen that Wigner functions are useful to visualize the phase-space properties of quantum states. Wigner functions display quadrature amplitudes, their fluctuations, and possible interferences.

3.2 Other quasiprobability distributions

In many respects the Wigner representation appears as the best compromise between a classical phase-space density and the correct quantum-mechanical behavior. The Wigner function generates the right marginal distributions and it obeys the overlap relation (3.20) for calculating expectation values in a classicallike fashion. Yet the Wigner function may be negative. Is there a way to define a strictly nonnegative quasiprobability distribution? Which other useful quasiprobability distributions can we define?

3.2.1 Q function

We may smooth the Wigner function $W(q, p)$ by convolving it with a Gaussian distribution having the same width as vacuum to obtain the *Q function*

$$Q(q,p) \equiv \frac{1}{\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} W(q',p')\exp[-(q-q')^2-(p-p')^2]\,dq'\,dp'. \tag{3.46}$$

What does this expression mean? We recall the overlap relation (3.20) and the formula (3.40) for Wigner functions of coherent states. We see immediately that the Q function gives simply the probability distribution for finding the coherent states $|\alpha\rangle$ with $\alpha = 2^{-1/2}(q + ip)$ in the state $\hat{\rho}$, because

$$\begin{aligned} Q(q,p) &= \frac{1}{2\pi}\mathrm{tr}\{\hat{\rho}|\alpha\rangle\langle\alpha|\} \\ &= \frac{1}{2\pi}\langle\alpha|\hat{\rho}|\alpha\rangle. \end{aligned} \tag{3.47}$$

(Note also that $\pi^{-1}\langle\alpha|\hat{\rho}|\alpha\rangle$ is frequently called the Q function.) Clearly, the Q function is nonnegative and normalized to unity, as is easily seen from the completeness relation (2.67) of the coherent states. Consequently, the Q function can be regarded as describing probability densities. In fact, we will consider in Chapter 6 a scheme to measure the Q function directly as a probability distribution. We also see that the negative regions of the Wigner function cannot be extended over areas significantly wider than $1/2$; otherwise, the Q function could be negative. (This property puts yet another constraint on Wigner functions.) Roughly speaking, the Gaussian smoothing (3.46) means taking the average of $W(q', p')$ values in a circle area around (q, p) with a radius given by the vacuum fluctuations. Because any negativities disappear after this procedure, they must be concentrated in small regions of the Wigner function. The resolution of these negative "probabilities" requires an accuracy in the order of the vacuum fluctuations.

The smoothing of the Wigner function is also clearly seen in the Fourier-transformed Q function $\tilde{Q}(u, v)$. In fact, we obtain from the definition (3.46)

$$\tilde{Q}(u, v) \equiv \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} Q(q, p)\exp(-iuq - ivp)\,dq\,dp \tag{3.48}$$

$$= \tilde{W}(u, v)\exp\left[-\frac{1}{4}(u^2 + v^2)\right]. \tag{3.49}$$

Because details of the Wigner function correspond to high-frequency components (u, v), these details are suppressed in the Q representation.

Eq. (3.49) reveals also another important property of the Q function. We use the formula (3.12) for the characteristic function $\tilde{W}(u, v)$ and obtain

$$\tilde{Q} = \mathrm{tr}\left\{\hat{\rho}\exp\left[-\frac{1}{4}(u^2 + v^2)\right]\exp(-iu\hat{q} - iv\hat{p})\right\} \tag{3.50}$$

or, introducing the complex notation $\beta = 2^{-1/2}(u + iv)$,

$$\tilde{Q} = \mathrm{tr}\left\{\hat{\rho}\exp\left(-\frac{1}{2}|\beta|^2\right)\exp(-i\hat{a}\beta^* - i\hat{a}^\dagger\beta)\right\} \tag{3.51}$$

because $\hat{a} = 2^{-1/2}(\hat{q} + i\hat{p})$. Finally, we employ the Baker–Hausdorff formula (2.55) and get

$$\tilde{Q} = \mathrm{tr}\{\hat{\rho}\exp(-i\hat{a}\beta^*)\exp\left(-i\hat{a}^\dagger\beta\right)\}. \tag{3.52}$$

Consequently,

$$\mathrm{tr}\{\hat{\rho}\hat{a}^\nu\hat{a}^{\dagger\mu}\} = i^{\nu+\mu}\frac{\partial^\nu}{\partial\beta^{*\nu}}\frac{\partial^\mu}{\partial\beta^\mu}\tilde{Q}\bigg|_{\beta=\beta^*=0}$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} Q(q, p)\alpha^\nu\alpha^{*\mu}\,dq\,dp \tag{3.53}$$

using the definition (3.48) of the Fourier-transformed Q function in the complex notation $\alpha = 2^{-1/2}(q + ip)$. Expectation values of the form $\mathrm{tr}\{\hat{\rho}\hat{a}^\nu\hat{a}^{\dagger\mu}\}$ are

called *antinormally ordered*. We have seen that we can express these quantities in terms of the Q function as if $\hat{a}$ and $\hat{a}^\dagger$ were classical amplitudes and not operators. We note, however, that this property relies critically on the ordering $\hat{a}^\nu\hat{a}^{\dagger\mu}$, and it is clearly lost when powers $(\hat{a}^\nu\hat{a}^{\dagger\mu})^\lambda$ are concerned. (Remember the discussion at the end of Section 3.1.2.)

3.2.2 *P function*

In the theory of photodetection (see for instance Ref. [187], Chap. 12), *normally ordered* expectation values $\mathrm{tr}\{\hat{\rho}\hat{a}^{\dagger\mu}\hat{a}^\nu\}$ play a distinguished role. How can we find the phase-space correspondence for normal ordering? We simply reverse the order of the exponentials $\exp(-i\hat{a}\beta^*)$ and $\exp(-i\hat{a}^\dagger\beta)$ in the expression (3.52) to define a new function

$$\tilde{P}(u, v) \equiv \mathrm{tr}\{\hat{\rho}\exp\left(-i\hat{a}^\dagger\beta\right)\exp(-i\hat{a}\beta^*)\} \tag{3.54}$$

with the convention $\beta = 2^{-1/2}(u + iv)$. Using the same arguments as in the previous subsection we see that the *P function*

$$P(q, p) \equiv \frac{1}{(2\pi)^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\tilde{P}(u, v)\exp(iuq + ivp)\,du\,dv \tag{3.55}$$

corresponds to the normal ordering

$$\mathrm{tr}\{\hat{\rho}\hat{a}^{\dagger\mu}\hat{a}^\nu\} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} P(q, p)\alpha^{*\mu}\alpha^\nu\,dq\,dp \tag{3.56}$$

with $\alpha = 2^{-1/2}(q + ip)$. Because normally ordered quantities are quite fundamental in quantum optics, the property (3.56) is one reason that the P function (also called the *Glauber–Sudarshan function* [106], [261]) is a rather popular phase-space distribution. Yet another reason is that the P function diagonalizes the density operator in terms of coherent states. To see this property, we argue along similar lines as in the previous subsection where we started from (3.49) and arrived at (3.52). Here we do the necessary algebra in reversed order – we start from the definition (3.54) and arrive via the Baker–Hausdorff formula (2.55) at

$$\tilde{W}(u, v) = \tilde{P}(u, v)\exp\left[-\frac{1}{4}(u^2 + v^2)\right]. \tag{3.57}$$

Consequently,

$$W(q, p) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} P(q_0, p_0)\frac{1}{\pi}\exp[-(q - q_0)^2 - (p - p_0)^2]\,dq_0\,dp_0. \tag{3.58}$$

We recall the formula (3.40) for Wigner functions of coherent states $|\alpha\rangle$ with $\alpha = 2^{-1/2}(q_0 + ip_0)$, and we use the general correspondence between the

Wigner function and the density matrix to obtain the famous result [261] (called the *optical equivalence theorem* [137])

$$\hat{\rho} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} P(q_0, p_0)|\alpha\rangle\langle\alpha|\, dq_0\, dp_0. \tag{3.59}$$

At first glance this formula appears as a representation of the quantum state in terms of a distribution of coherent states, that is, as a statistical mixture of classical amplitudes. This is impossible! There is no way to represent a pure state $|\psi\rangle$ as a mixture of coherent states, unless $|\psi\rangle$ itself is a coherent state. Yet the algebra to arrive at the result (3.59) is correct. What is wrong? The answer is that the P function might be very ill-behaved. For instance, we see from Eq. (3.57) that the Wigner function is a smoothed P function. Because the Wigner function can display negative "probabilities" the P function might behave even worse, that is, it might be negative or it might not even exist as a tempered distribution. (States having such P functions are called *nonclassical states*. See the discussion at the beginning of Section 2.2.3.) Because the P function is such a delicate mathematical construction, there is no practical way to reconstruct it from experimental data.

3.2.3 *s*-parameterized quasiprobability distributions

We may also convolve the Wigner function with Gaussian distributions having a width different from what would correspond to the vacuum noise. In this way we obtain a whole family of distributions called the *s-parameterized quasiprobability distributions* $W(q, p; s)$ [51], [52], [156]. First, we define the characteristic functions

$$\tilde{W}(u, v; s) \equiv \tilde{W}(u, v)\exp\left[\frac{s}{4}(u^2 + v^2)\right]. \tag{3.60}$$

For historical reasons [51], [52] the real parameter s happens to be negative when the Wigner function is smoothed. We see from definition (3.60) that in this case the distribution is indeed suppressing high frequencies (u, v) describing details in the Wigner function. The s-parameterized quasiprobability distributions are obtained from the characteristic function via inverse Fourier transformation

$$W(q, p; s) \equiv \frac{1}{(2\pi)^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \tilde{W}(u, v; s)\exp(iuq + ivp)\, du\, dv. \tag{3.61}$$

Obviously, all previously studied quasiprobability distributions are included in this family of functions because they correspond to the parameters

$$s = \begin{cases} +1: & P \text{ function,} \\ 0: & \text{Wigner function,} \\ -1: & Q \text{ function,} \end{cases} \tag{3.62}$$

respectively. In this way the defined distributions interpolate between the P, the Wigner, and the Q function. The range of s, however, is the whole real axis. Note that it is also possible to define quasiprobability distributions corresponding to complex s parameters [300].

Let us study some general properties of s-parameterized quasiprobability distributions. Of course, they are normalized to unity because

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} W(q, p; s)\, dq\, dp = \tilde{W}(0, 0; s) = 1. \tag{3.63}$$

We obtain from the obvious relation

$$\tilde{W}(u, v; s) = \tilde{W}(u, v; s')\exp\left[\frac{1}{4}(s - s')(u^2 + v^2)\right] \tag{3.64}$$

the formula

$$W(q, p; s) = \frac{1}{\pi(s' - s)}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} W(q', p'; s') \times \exp\left[-\frac{(q - q')^2 + (p - p')^2}{(s' - s)}\right] dq'\, dp', \tag{3.65}$$

provided that $s < s'$ so that the integral converges. This relation shows that there is a smoothing hierarchy among the s-parameterized quasiprobability distributions. The smaller the parameter s is, the more the distribution is smoothed. Moreover, the marginal distributions $\mathrm{pr}(q, \theta; s)$ of smoothed Wigner functions $(s < 0)$ are smoothed accordingly, that is,

$$\mathrm{pr}(q, \theta; s) \equiv \int_{-\infty}^{+\infty} W(q\cos\theta - p\sin\theta, q\sin\theta + p\cos\theta; s)\, dp \tag{3.66}$$

$$= (\pi|s|)^{-1/2}\int_{-\infty}^{+\infty} \mathrm{pr}(q', \theta)\exp[-|s|^{-1}(q - q')^2]\, dq', \tag{3.67}$$

because the Wigner function has the quantum-mechanically correct marginals $\mathrm{pr}(q, \theta)$. Additionally, the overlap relation (3.20) must be modified for s-parameterized quasiprobability distributions because

$$\begin{aligned} \mathrm{tr}\{\hat{F}_1\hat{F}_2\} &= 2\pi\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} W_1(q, p)W_2(q, p)\, dq\, dp \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \tilde{W}_1(u, v)\tilde{W}_2(-u, -v)\, du\, dv \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \tilde{W}_1(u, v; s)\tilde{W}_2(-u, -v; -s)\, du\, dv. \end{aligned} \tag{3.68}$$

Consequently, we obtain

$$\mathrm{tr}\{\hat{F}_1\hat{F}_2\} = 2\pi\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} W_1(q, p; s)W_2(q, p; -s)\, dq\, dp \tag{3.69}$$

and in particular

$$\mathrm{tr}\{\hat{\rho}\hat{F}\} = 2\pi \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} W(q,p;s)W_F(q,p;-s)\,dq\,dp. \tag{3.70}$$

This relation shows that a smoothing of the quasiprobability distribution must be compensated for by an enhancement of the resolution of the filter function $W_F(q,p)$ to calculate expectation values, and vice versa. This procedure may cause significant problems because it requires extremely high accuracy for $W(q,p;s)$ and it may involve singular filter functions $W_F(q,p;s)$. We see that the price to be paid for having a nonnegative quasiprobability distribution is the introduction of additional noise in practical applications. This noise appears in the smoothing of the marginal distributions, and it must be compensated for by enhancing filter functions to correctly predict observable quantities. Finally, we also note that a certain *s-ordering* of operators [52] can be defined to calculate expectation values. However, apart from the normal ordering for the P function, the symmetric ordering for the Wigner representation, and the antinormal ordering corresponding to the Q function, these ordering procedures are involved. The reader is referred to the comprehensive articles by Cahill and Glauber [51], [52] for the details.

3.3 Examples

How do Q functions look? How smoothed are they compared with Wigner functions? How singular can a P function be? Let us study some examples. The simplest candidate to consider theoretically is a Fock state $|n\rangle$. We see immediately from formula (3.47) and the Poissonian photon statistics (2.64) of a coherent state that the Q function of a Fock state is given by

$$\begin{aligned} Q(q,p) &= \frac{1}{2\pi}|\langle\alpha\,|\,n\rangle|^2 \\ &= \frac{1}{2\pi n!}\exp(-|\alpha|^2)|\alpha|^{2n} &(3.71)\\ &= \frac{1}{2\pi n!}\exp\left[-\frac{1}{2}(q^2+p^2)\right]\left(\frac{q^2+p^2}{2}\right)^n. &(3.72)\end{aligned}$$

According to the Gaussian approximation for the Poissonian distribution [279], [241] we can approximate (3.72) for large quantum numbers and get

$$Q_n \sim \frac{1}{2\pi^{3/2}r}\exp[-(r-r_n)^2] \tag{3.73}$$

with

$$r = \sqrt{q^2+p^2} \tag{3.74}$$

and the Bohr–Sommerfeld radius [79]

$$r_n = \sqrt{2n+1}. \tag{3.75}$$

We see that the Q function of a Fock state describes a ring with the Bohr–Sommerfeld radius r_n in phase space. This illustrates that Fock states are typical particlelike states containing exactly n energy quanta and showing no wavelike phase dependence. The P function for a Fock state is obtained from Eq. (3.72) by Fourier transformation according to the general relations (3.60) and (3.62). The result

$$P = \frac{1}{n!}\exp\left(\frac{q^2+p^2}{2}\right)\left[\frac{1}{2}\left(\frac{\partial^2}{\partial q^2}+\frac{\partial^2}{\partial p^2}\right)\right]^n \delta(q)\delta(p) \tag{3.76}$$

indicates clearly that Fock states are indeed nonclassical because their P functions contain derivatives of the two-dimensional delta function $\delta^2(\alpha)$. (The only exception is obviously the vacuum state with $n=0$.) This example illustrates the mathematical subtleties involved in the P representation.

Let us consider another example, a thermal state, with density operator

$$\hat{\rho} = (1-e^{-\beta})\sum_{n=0}^{\infty}|n\rangle\langle n|e^{-n\beta}. \tag{3.77}$$

Here β denotes the ratio $\hbar\omega/k_BT$ of the energy $\hbar\omega$ and the temperature T. (As usual k_B denotes Boltzmann's constant.) To justify the formula (3.77) we recall that in thermal equilibrium the density operator must be diagonal in the energy representation and that photons obey the Bose statistics. We use expression (3.71) to calculate the Q function of the thermal state (3.77)

$$\begin{aligned} Q(q,p) &= (1-e^{-\beta})\frac{1}{2\pi}\exp(-|\alpha|^2)\sum_{n=0}^{\infty}\frac{1}{n!}(|\alpha|^2e^{-\beta})^n \\ &= \frac{1}{2\pi}(1-e^{-\beta})\exp[-|\alpha|^2(1-e^{-\beta})] &(3.78)\\ &= \frac{1}{2\pi}(1-e^{-\beta})\exp\left[-\frac{1}{2}(q^2+p^2)(1-e^{-\beta})\right]. &(3.79)\end{aligned}$$

The Q function is a Gaussian distribution centered at the origin in phase space. In the limiting case of vanishing temperature, we obtain the Q function $Q_0(q,p)$ of a vacuum, whereas for finite temperature the Gaussian distribution (3.79) is accordingly broader. This difference illustrates the additional fluctuations involved in a thermal state. Using Fourier transformation we obtain from the Q function (3.79) the Wigner function for a thermal state

$$W(q,p) = \frac{1}{\pi}\tanh(\beta/2)\exp[-(q^2+p^2)\tanh(\beta/2)]. \tag{3.80}$$

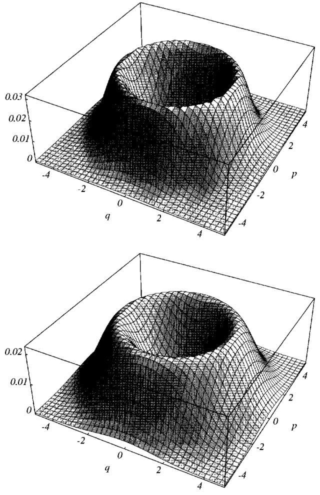

Fig. 3.7. Q functions of a Fock state (top), Eq. (3.72), and of a phase-randomized coherent state (bottom), Eqs. (3.90) and (3.75), with $n = 4$. Although the states are quite different, the Q functions are similar.

Like the Q function, the Wigner function displays the additional thermal fluctuations as well. Note that formula (3.80) also reveals the P function for the thermal density operator (3.77) by Fourier transformation

$$P(q, p) = \frac{1}{\pi}(e^{\beta} - 1)\exp[-(q^2 + p^2)(e^{\beta} - 1)]. \tag{3.81}$$

The P function of a thermal state is a well-behaved positive function that can be rightfully regarded as a probability distribution. In this sense thermal states are classical states. According to Eq. (3.59) the P function diagonalizes the density operator in terms of coherent states. So instead of seeing the thermal state as a statistical mixture of nonclassical Fock states, we may unravel the thermal density operator (3.77) into a Gaussian distribution of coherent states, that is, into an incoherent mixture of states with well-defined amplitudes and phases. In this way we find a satisfying physical interpretation of thermal states and simultaneously a good example to demonstrate the general ambiguity in unraveling a mixed-state density operator. See Section 1.3.4.

Formula (3.80) reveals the Wigner function $W_n(q, p)$ of Fock states as well. We expand $W(q, p)$ in terms of $e^{-\beta}$

$$W(q, p) = (1 - e^{-\beta})\sum_{n=0}^{\infty} W_n(q, p)e^{-n\beta} \tag{3.82}$$

with

$$W_n(q, p) = \frac{(-1)^n}{\pi}\exp(-q^2 - p^2)L_n(2q^2 + 2p^2). \tag{3.83}$$

Here the $L_n(q)$ denote the Laguerre polynomials, and we have utilized their relation

$$\sum_{n=0}^{\infty} L_n(q)z^n = (1 - z)^{-1}\exp[qz(z - 1)^{-1}]. \tag{3.84}$$

See Ref. [89], Eq. 10.12(17). Because the thermal density operator (3.77) is expanded in the same way as the expression (3.82), the $W_n(q, p)$ must be indeed the Wigner functions for the Fock states $|n\rangle$. Figure 3.8 shows that in contrast to the Q function, the Wigner function for a Fock state displays a "wavy sea" of rings in the area enclosed by the Bohr–Sommerfeld band [79]. This feature illustrates again that Fock states are clearly nonclassical. Note that the "wavy sea" is necessary to guarantee the orthogonality of the Fock states because the overlap of two Wigner functions $W_n(q, p)$ and $W_{n'}(q, p)$ must vanish according to formula (3.23). The transition to the Q function, however, smoothes out the waves, and only the Bohr–Sommerfeld ring at $(2n + 1)^{1/2}$ remains. This result shows strikingly that the Q function displays far less signature of a quantum state in phase space than does the Wigner function.

We can see this another way. Significantly different quantum states may create similar Q functions. Given a picture of a Q function, it may be difficult

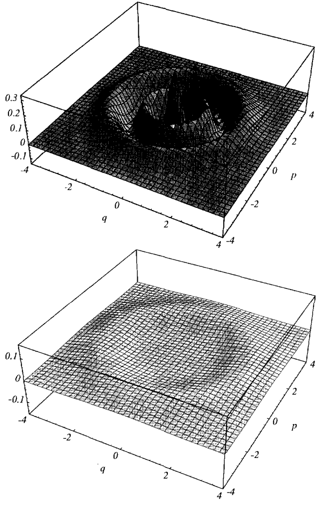

Fig. 3.8. Wigner function of a Fock state with $n = 4$ (top), Eq. (3.83), compared with the Q function (bottom), Eq. (3.72). The plot range for the Q function was set to half of the range for the Wigner function to visualize the differences between the Q and the Wigner representation.

to infer the state behind the picture. Probably the best example to demonstrate this is a Schrödinger-cat state

$$|\psi\rangle \propto |\alpha_0\rangle + |-\alpha_0\rangle \tag{3.85}$$

(we omit the normalization factor). Using the scalar-product (2.66) of coherent states, we obtain immediately from formula (3.47) the Q function

$$\begin{aligned} Q(\alpha) \propto\ & \exp(-|\alpha-\alpha_0|^2) + \exp(-|\alpha+\alpha_0|^2) \\ & + 2\exp(-|\alpha|^2 - |\alpha_0|^2)\cos[2\mathrm{Im}(\alpha^*\alpha_0)]. \end{aligned} \tag{3.86}$$

All that is left from the beautiful quantum-interference structure clearly displayed in the Wigner function (3.42) is an exponentially small bump proportional to $\exp(-|\alpha_0|^2)$. The more macroscopic the quantum superposition (3.85) is, the smaller is this term. If we neglect the little hump we obtain the Q function of the incoherent mixture

$$\hat{\rho} = \frac{1}{2}(|\alpha_0\rangle\langle\alpha_0| + |-\alpha_0\rangle\langle-\alpha_0|). \tag{3.87}$$

The Q function cannot clearly discriminate between macroscopic superpositions and statistical mixtures, that is, between the classical *either* α_0 *or* $-\alpha_0$ and the quantum-mechanical α_0 *as well as* $-\alpha_0$.

Another example is a phase-randomized coherent state having the density operator

$$\hat{\rho} = \int_0^{2\pi} |2^{-1/2} r_n \exp(\mathrm{i}\phi)\rangle\langle 2^{-1/2} r_n \exp(\mathrm{i}\phi)| \frac{d\phi}{2\pi} \tag{3.88}$$

with the Bohr–Sommerfeld radius r_n defined in (3.75). This state creates almost the same picture as a Fock state in the Q representation. The Q function of a coherent state is

$$Q(q,p) = \frac{1}{2\pi}\exp\left[-\frac{1}{2}(q-q_0)^2 - \frac{1}{2}(p-p_0)^2\right], \tag{3.89}$$

as is easily obtained from the Wigner function (3.40) or, alternatively, from the scalar product (2.66). Consequently, the Q function of the phase-randomized coherent state (3.88) is given by

$$Q(r) = \frac{1}{2\pi}\exp\left[-\frac{1}{2}(r^2 + r_n^2)\right] I_0(r r_n) \tag{3.90}$$

using Ref. [225], Eq. 2.5.40.3, to perform the ϕ integration. Here I_0 denotes the zeroth modified Bessel function of the first kind, and the radii r and r_n have been defined in Eqs. (3.74) and (3.75). According to the asymptotic behavior of the Bessel functions for large arguments, expression (3.90) tends rapidly to the approximation

$$Q \sim \frac{1}{(2\pi)^{3/2}(r_n r)^{1/2}} \exp[-(r-r_n)^2] \tag{3.91}$$

for large quantum numbers. See Ref. [89], Eq. 7.13.1(5). Similar to Fock states, the Q functions of phase-randomized coherent states describe rings in phase space. The only difference is that the rings are broader by a factor of $2^{1/2}$. Yet the states and their physical properties are significantly different. For instance, the photon statistics of phase-randomized coherent states is Poissonian (2.64). (Phase changes do not affect the photon statistics.) In the limit of large intensities the photon distribution (2.64) becomes extremely broad because the variance equals the mean for Poissonian statistics. Fock states, however, always have a precise photon number. Paradoxically, in the region where the photon distributions of phase-randomized coherent states and of Fock states are vastly different, their Q functions are very similar. The Wigner functions, however, differ significantly.

Nevertheless, both the Q function and the P function are mathematically equivalent to any other representation for quantum states, and we are entitled to use this equivalence in tricks to derive theoretical relations. On the other hand, when the Q function is numerically or experimentally given, the retrieval of details hidden in the Q function (but clearly displayed in the Wigner function) takes significant effort in precision. In any case, we must perform a deconvolutionlike procedure that is typically delicate. This fact motivates the conclusion that experimental efforts should be aimed at measuring the Wigner function rather than the Q function to determine the quantum state. The measurement or even the reconstruction of the P function is clearly beyond feasibility, because this mathematical object might be ill-behaved, as we have seen in the case of a Fock state.

3.4 Further reading

Apart from Wigner's Wigner function [290] defined in the phase space of position and momentum, other possible Wigner functions exist for different systems. For instance, spin systems may be described by continuous Wigner functions. See G.S. Agarwal [3]; J.P. Dowling, G.S. Agarwal, and W.P. Schleich [80]; M.O. Scully [248]; L. Cohen and M.O. Scully [57]; M.O. Scully and K. Wódkiewicz [249]; M.O. Scully, H. Walther, and W.P. Schleich [250]; K. Wódkiewicz [295]; and J.C. Várilly and J.M. Gracia-Bondia [277].

The discrete Wigner functions are another intriguing class of quasiprobability distributions for finite-dimensional systems. See the interesting paper by W.K. Wootters [298] for prime-dimensional state spaces and the extension to odd-dimensional systems by O. Cohendet, Ph. Combe, M. Sirugue, and M. Sirugue-Collin [61], [62]. Wigner functions for even dimensions are a bit odd, and they, together with the odd-dimensional ones, have been considered in a brief communication [165] and in a detailed paper [170]. Moreover, a discrete Q function has been defined by T. Opatrný, V. Bužek, J. Bajer, and G. Drobný [203] and extended to discrete s-parameterized quasiprobability distributions by T. Opatrný, D.-G. Welsch, and V. Bužek [204].

Also, Wigner functions for angular momentum and phase have been constructed by N. Mukunda [192] and J.P. Bizarro [31]. Wigner functions for photon-number and quantum-optical phase have been given by W.P. Schleich, R.J. Horowicz, and S. Varro [242]; J.A. Vaccaro and D.T. Pegg [271]; A. Lukš and V. Peřinová [183]; and, finally, in a communication [272] and in a comprehensive paper [274] by J.A. Vaccaro.